

# Casimir Effect For a Scalar Field via Krein Quantization

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## Abstract

In this paper, we apply the indefinite metric field (Krein) quantization method to study the Casimir effect for a massless scalar field on a sphere with Dirichlet boundary condition. In this method negative norm (un-physical) states automatically regularize the theory. Having understood that these states are un-physical they are only used as mathematical tools for renormalizing the theory and then one can get rid of them by imposing some proper physical conditions.

## 1 Introduction

The study of states with negative norms has a long story dating back to Dirac's work in 1942 [1]. Followed by Gupta and Bleuler in 1950, such states were used to remove infrared divergence in QED [2]. In this way it is proved that quantization of the minimally coupled massless scalar field in de Sitter space can be done covariantly by the help of negative norms [3, 4]. In other words, due to the famous 'zero-mode' problem [5, 3], one can not define a proper de Sitter invariant vacuum state with only positive norms, so a Gupta-Bleuler type construction based on Krein space structure is needed [3]. Furthermore, in calculating graviton propagator in de Sitter background in the linear approximation, pathological behavior for largely separated points is removed if one calculates it in Krein space [3, 4, 6]. [This result is in agreement with other works since previously it was shown that infrared divergence of graviton propagator in one loop approximation do not appear in an effective way as a physical quantity because of gauge dependency [7, 8, 9].]

Quantization in Krein space has some other advantages for example, the vacuum energy vanishes without any need of reordering the terms and infinite term does not appear in the calculation of the expectation value of  $T_{\mu\nu}$ , which is independent of the curvature. Furthermore, in the interaction QFT this method works well in studying (in fact automatically removing) the singular behaviors of the Green's function at short relative distances (ultraviolet divergence) except the light cone singularity [10]. Recently, it was shown that in Krein quantization method, one-loop effective action for QED is automatically regularized [11].

In Ref. [12], through this approach, the Casimir force between parallel plates in flat space has been calculated which simply gave the correct result. In this paper, we study this effect in Krein space, for a  $S^2$ -sphere with the Dirichlet boundary condition.

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The layout of the paper is as follows: The method of Krein space is briefly reviewed and also the strategy against the negative norm states and getting rid of them are all discussed in section 2. In section 3, through this method, we calculate the Casimir energy and as it is shown, negative norm states automatically regularize the theory and after renormalization the results are the same as the previous related works. A brief discussion is done as a conclusion in section 4. Finally, we have enclosed the paper with some details of mathematical calculations in two appendices.

## 2 Casimir effect and Krein quantization

An attractive force between two uncharged, parallel conducting plates was predicted by Casimir [13] hence this is called Casimir effect. This effect can be regarded as the physical manifestation of the zero point energy and its quantum nature can be understood by the zero point fluctuation of the quantum field which has to satisfy boundary conditions. A great deal of techniques have been proposed for studying the Casimir effect, e.g., direct mode summation [14], zeta function regularization [15], Green's functions [16] and stress-tensor methods [17]. Here we use the Krein space field quantization method, first let us recall the method.

### ■ The method:

Hilbert space is built by a set of modes with positive norms:

$$\mathcal{H} = \left\{ \sum_{k \geq 0} \alpha_k \phi_k; \sum_{k \geq 0} |\alpha_k|^2 < \infty \right\}, \text{ with } (\phi_1, \phi_2) > 0,$$

Krein space is defined as a direct sum of an Hilbert space and an anti-Hilbert space (negative inner product space):

$$\mathcal{K} = \mathcal{H} \oplus \bar{\mathcal{H}},$$

where  $\bar{\mathcal{H}}$  stands for the anti-Hilbert space. Note that due to the indefinite inner product space, some states are allowed to have negative norm. Let us illustrate Krein quantization by giving a simple example. A free massless scalar field  $\phi(\vec{x}, t)$  which satisfies the Klein-Gordon equation

$$\square \phi(x) = \left( \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \phi(\vec{x}, t) = 0, \quad (2.1)$$

has two sets of solutions:

$$u_P(\vec{k}, t, \vec{x}) = \frac{e^{i\vec{k} \cdot \vec{x} - i\omega t}}{\sqrt{(2\pi)2\omega}}, \text{ and } u_N(\vec{k}, t, \vec{x}) = \frac{e^{-i\vec{k} \cdot \vec{x} + i\omega t}}{\sqrt{(2\pi)2\omega}}, \quad (2.2)$$

where the inner product is defined by

$$\langle \phi_1, \phi_2 \rangle = i \int_{t=\text{const}} dx (\phi_1^* \partial_t \phi_2 - \phi_2 \partial_t \phi_1^*), \quad (2.3)$$

and these modes are normalized by the following relations

$$\begin{aligned} \langle u_P(\vec{k}, \vec{x}, t), u_P(\vec{k}', \vec{x}, t) \rangle &= \delta(\vec{k} - \vec{k}'), \\ \langle u_N(\vec{k}, \vec{x}, t), u_N(\vec{k}', \vec{x}, t) \rangle &= -\delta(\vec{k} - \vec{k}'), \\ \langle u_P(\vec{k}, \vec{x}, t), u_N(\vec{k}', \vec{x}, t) \rangle &= 0. \end{aligned} \quad (2.4)$$

The states with negative norm have negative energy as well (because of the wrong sign of  $kx$  in  $u_N$ ) in fact, they are un-physical states. In ordinary quantum field theory in Hilbert space these un-physical

states are ignored, however, in Krein space they are used to build the quantum field. It is important to note that these un-physical states must be ruled out by imposing some conditions. The proper condition can be stated as follows:

- These un-physical states only appear in the internal Feynman diagrams not external ones this guarantees that they do not propagate in the physical world. The direct consequence is that the physical boundary conditions have no effect on these states<sup>1</sup>, in other words they are only used as mathematical tools in the theory [10, 11].
- The unitarity of the theory is survived by renormalizing the elements of the  $S$  matrix (probability amplitude) as [18]

$$S_{if} = \frac{\langle \text{physical states, in} | \text{physical states, out} \rangle}{\langle 0, \text{in} | 0, \text{out} \rangle}, \quad (2.5)$$

where only the physical states are taken into account. This condition eliminates the un-physical states from the disconnected parts.

Let us return to our example in which the field operator in Krein space is defined by [10]

$$\phi(t, \vec{x}) = \frac{1}{\sqrt{2}} (\phi_P(t, \vec{x}) + \phi_N(t, \vec{x})), \quad (2.6)$$

where both  $\phi_P$  and  $\phi_N$  have been used and are defined by

$$\begin{aligned} \phi_P(t, \vec{x}) &= \int d\vec{k} [a(\vec{k}) u_P(\vec{k}, t, \vec{x}) + a^\dagger(\vec{k}) u_P^*(\vec{k}, t, \vec{x})], \\ \phi_N(t, \vec{x}) &= \int d\vec{k} [b(\vec{k}) u_N(\vec{k}, t, \vec{x}) + b^\dagger(\vec{k}) u_N^*(\vec{k}, t, \vec{x})]. \end{aligned} \quad (2.7)$$

The quantum theory is identified by defining the vacuum state as:

$$\begin{aligned} a(\vec{k})|\Omega\rangle &= 0, \quad a^\dagger(\vec{k})|\Omega\rangle = |1_k\rangle, \\ b(\vec{k})|\Omega\rangle &= 0, \quad b^\dagger(\vec{k})|\Omega\rangle = |\overline{1_k}\rangle, \end{aligned} \quad (2.8)$$

where  $|\overline{1_k}\rangle$  is an un-physical state. A significant difference with the standard QFT, which is based on the canonical commutation relation, lies in the requirement of the following commutation relations:

$$\begin{aligned} [a(\vec{k}), a^\dagger(\vec{k}')] &= \delta(\vec{k} - \vec{k}'), \quad [a(\vec{k}), a(\vec{k}')] = [a^\dagger(\vec{k}), a^\dagger(\vec{k}')] = 0, \\ [b(\vec{k}), b^\dagger(\vec{k}')] &= -\delta(\vec{k} - \vec{k}'), \quad [b(\vec{k}), b(\vec{k}')] = [b^\dagger(\vec{k}), b^\dagger(\vec{k}')] = 0, \\ [a(\vec{k}), b(\vec{k})] &= [a(\vec{k}), b^\dagger(\vec{k})] = [a^\dagger(\vec{k}), b(\vec{k})] = [a^\dagger(\vec{k}), b^\dagger(\vec{k})] = 0. \end{aligned} \quad (2.9)$$

With the field operator (2.6), let us carry out the calculations in the presence of the boundary condition.

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<sup>1</sup>In the photon field case, these un-physical states may conflict with the longitudinal modes whose norm is negative. Noting that the second one has positive energy and has been affected by the boundary condition due to the gauge covariance [21, 20], however here as mentioned, un-physical states are those states whose norms and energies are negative.

### 3 calculations:

The study of Casimir effect in sphere has a quite rich history. The usual way is calculating the difference in zero-point energy between two configurations, namely, the system consists of a large sphere enclosing the quantization universe and a small concentric sphere with radius  $a$  [21]. However in this work, only oscillation modes inside the sphere are considered, thus the calculations are less complicated. In spherical coordinates there are two sets of solutions and only states whose norm is positive are affected by the (Dirichlet) boundary condition. For a sphere of radius  $a$  one can write:

$$E_{(sphere)}^{Krein} = \sum_{l=0}^{\infty} (l + \frac{1}{2}) \sum_{n=0}^{\infty} \omega_{nl} - \frac{a^3}{3\pi} \int_0^{\infty} K^3 dK, \quad (3.1)$$

where the eigenfrequencies  $\omega_{nl}$  are determined by the spherical Bessel function  $j_l(\omega a) = 0$ , [ $j_l(z) = \sqrt{\frac{\pi}{2z}} J_{l+1/2}(z)$ ] with  $l = 0, 1, 2, \dots$ . The second term at the right hand side of Eq. (3.1) appears due to the un-physical states and by straightforward algebra it is easy to show that it can be written as (Appendix A)

$$\frac{a^3}{3\pi} \int_0^{\infty} K^3 dK \simeq \frac{\pi}{a} \sum_{l=0}^{\infty} (l + \frac{1}{2}) \sum_{n=0}^{\infty} (n + \frac{1}{2}). \quad (3.2)$$

After making use of the Hurwitz  $\zeta$ -function<sup>2</sup> and doing some algebra, one obtains:

$$E_{(sphere)}^{Krein} = \sum_{l=0}^{\infty} (l + \frac{1}{2}) \sum_{n=0}^{\infty} (\omega_{nl} - \bar{\omega}_{nl}), \quad (3.3)$$

$$\bar{\omega}_{nl} \equiv \frac{\pi}{a} [(n + \frac{1}{2}) + \frac{l+1}{2}],$$

where  $\bar{\omega}_{nl}$  are determined by  $\lim_{a \rightarrow \infty} j_l(\bar{\omega}_{nl} a) = 0$ . As it is clear from (3.3), the role of the negative norms can be understood as a regularizer of the theory.

It is suitable to write  $E_{(sphere)}^{Krein} = \sum_{l=0}^{\infty} E_l^{Krein}$  by defining

$$E_l^{Krein} \equiv (l + \frac{1}{2}) \sum_{n=0}^{\infty} (\omega_{nl} - \bar{\omega}_{nl}). \quad (3.4)$$

At this stage, we use Cauchy's theorem to convert the summation (3.4) in to the integral form which reads as

$$E_l^{Krein} = \frac{l+1/2}{2\pi i} \left( \oint_C dz z \frac{d}{dz} \ln \frac{f(z, a)}{f(z, a \rightarrow \infty)} \right), \quad (3.5)$$

where  $f(z, a)$  is defined by<sup>3</sup>

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<sup>2</sup>Hurwitz  $\zeta$ -function is defined by [22, 23]

$$\zeta(z, q) = \sum_{n=0}^{\infty} \frac{1}{(q+n)^z},$$

where for  $q = 1/2$  is related to the Riemann  $\zeta$ -function  $\zeta(z, 1/2) = (2^z - 1)\zeta(z)$ . Therefore, it follows that

$$\frac{(l+1)}{2} \sum_{n=0}^{\infty} (n + \frac{1}{2})^0 = \frac{(l+1)}{2} \zeta(0, 1/2) = 0.$$

<sup>3</sup>It should be noted that  $J_{\nu}(z)$ ,  $\nu > -1$ , has no complex roots [24]. Therefore according to Cauchy's integral theorem (Eq. (3.5)),  $f(z, a) = J_{\nu}(a z e^{i\theta})$  or  $J_{\nu}(a z e^{i|\theta|})$  leads to same result for  $E_l^{Krein}$ .

$$f(z, a) \equiv J_\nu(a \mathbf{z} e^{i|\theta|}), \quad \nu = l + 1/2 = 1/2, 3/2, \dots \quad (3.6)$$

("z" and "θ" are respectively the modulus and argument of  $z$ .) The contour of integration consists of two parts. The first part is a counterclockwise semicircular  $C_\Lambda$ , centered at the origin in the right half-plane and having a radius  $\Lambda$  large enough to enclose all the poles of  $f(z, a)$ . The second part of the contour is the straight line from  $(0, i\Lambda)$  to  $(0, -i\Lambda)$  on the imaginary axis. Note that  $z$  in (3.5) can be taken as  $\lim_{m \rightarrow 0} \sqrt{z^2 + m^2}$ , therefore in the integral along the segment  $(i\Lambda, -i\Lambda)$  we can integrate once by parts and the nonintegral terms being cancelled [25]. Therefore, Eq. (3.5) becomes

$$E_l^{Krein} = \frac{l + 1/2}{\pi} \int_0^\infty dy \ln \frac{f(iy, a)}{f(iy, a \rightarrow \infty)} + \frac{l + 1/2}{2\pi i} \int_{C_\Lambda(-\frac{\pi}{2}, \frac{\pi}{2})} z d \ln \frac{f(z, a)}{f(z, a \rightarrow \infty)}, \quad (3.7)$$

where we have (Appendix A)

$$\frac{f(iy, a)}{f(iy, a \rightarrow \infty)} = \frac{J_\nu(iya)}{\lim_{a \rightarrow \infty} J_\nu(iya)} = \frac{I_\nu(ya)}{\lim_{a \rightarrow \infty} I_\nu(ya)} = \sqrt{2\pi a y} e^{-ay} I_\nu(ya), \quad (3.8)$$

therefore one obtains

$$E_l^{Krein} = \frac{l + 1/2}{\pi} \int_0^\infty dy \ln (\sqrt{2\pi a y} e^{-ay} I_\nu(ya)) + \frac{l + 1/2}{2\pi i} \int_{C_\Lambda(-\frac{\pi}{2}, \frac{\pi}{2})} z d \ln \frac{f(z, a)}{f(z, a \rightarrow \infty)}, \quad (3.9)$$

Eq. (3.9) after relatively simple calculations, can be written as

$$\begin{aligned} E_l^{Krein} &= \frac{l + 1/2}{\pi} \int_0^\infty dy \ln(2ay K_\nu(ay) I_\nu(ay)) \\ &+ \frac{l + 1/2}{\pi} \int_0^\infty dy \ln \frac{\lim_{a \rightarrow \infty} K_\nu(ay)}{K_\nu(ay)} + \frac{l + 1/2}{2\pi i} \int_{C_\Lambda(-\frac{\pi}{2}, \frac{\pi}{2})} z d \ln \frac{f(z, a)}{f(z, a \rightarrow \infty)}. \end{aligned} \quad (3.10)$$

$I_\nu(z)$  and  $K_\nu(z)$  are modified Bessel functions. In Appendix B, it is shown that the sum of two last terms in (3.10) vanishes, then we get

$$\begin{aligned} E_l^{Krein} &= \frac{l + 1/2}{\pi} \int_0^\infty dy \ln(2ay K_\nu(ay) I_\nu(ay)), \\ &= \frac{l + 1/2}{a\pi} \int_0^\infty dy \ln(2y K_\nu(y) I_\nu(y)). \end{aligned} \quad (3.11)$$

For large  $\nu$  the above integral behaves as follows

$$E_l^{Krein} \simeq \frac{1}{a} \left( -\frac{\nu^2}{2} - \frac{1}{128} + \frac{35}{32768\nu^2} + \mathcal{O}(\nu^{-3}) \right). \quad (3.12)$$

Substituting (3.12) in (3.3), because of the first two terms, leads to divergence, but it can be handled easily by rewriting (3.3) as [25]

$$\begin{aligned} E_{(sphere)}^{Krein} &= \sum_{l=0}^\infty \tilde{E}_l^{Krein} - \frac{1}{2a} \sum_{l=0}^\infty (l + 1/2)^2 - \frac{1}{128a} \sum_{l=0}^\infty (l + 1/2)^0, \\ &= \sum_{l=0}^\infty \tilde{E}_l^{Krein} - \frac{1}{2a} \zeta(-2, 1/2) - \frac{1}{128a} \zeta(0, 1/2) = \sum_{l=0}^\infty \tilde{E}_l^{Krein}, \end{aligned} \quad (3.13)$$

where

$$\tilde{E}_l^{Krein} \equiv E_l^{Krein} + \frac{(l+1/2)^2}{2a} + \frac{1}{128a} \simeq \frac{1}{a} \frac{35}{32768\nu^2},$$

note that  $\zeta(-2, 1/2) = \zeta(0, 1/2) = 0$  [22, 23]. Therefore, the last sum in (3.13) obviously converges and by the means of numerical integration, we obtain

$$E_{(sphere)}^{Krein} = \frac{1}{a} 0.002819..., \quad (3.14)$$

which is the Casimir energy in the Sphere [26, 27].

## 4 Conclusion

With the implementation of the Krein space quantization method, it has been shown that one obtains a correct result for the Casimir effect in two parallel plates [12]. In this paper we extended this method to study the Casimir energy for a  $S^2$ -sphere which automatically gives us its regularized form and accordingly the renormalization procedure was done. Based on previous works and this study, it seems quantization in Krein space either can remove the infinity or at least, by regularization makes it easy to handel.

Interestingly, it has been shown that Casimir effect depends strongly on the geometry of the space-time or on the imposed boundary conditions [as well as on the the dimensions of the space-time [27]], for example, Boyer [28] showed the Casimir energy for the sphere is positive which implies a repulsive force for this configuration rather than anticipated attractive force. But to our knowledge up to now there is not an explicit analysis of being repulsive or attractive in the literature. It may be useful considering the role of negative norms in this area. And also studying the Casimir effect in de Sitter-like geometries *e.g.*,  $R^1 \times S^3$ , and its connection to the present accelerated universe is interesting. Of course such relations have been considered in the literature (for example in [29]), however, consideration in Krein space makes it easy to analyze the connection between large scale expansion and the geometry.

Furthermore, in quantum field theory, consideration of quantum effects on physical quantities is understood by the means of expectation values which are related to the Green's functions. It is well known that in calculating Green's functions some infinities are appeared. On the other hand, it has been proved in the Krein space quantization, by considering quantum metric fluctuations, the divergences of the Green's function in QED are removed [30, 31]. This may shed some light on better understanding of the quantization of massless higher spin fields in curved space-time.

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## A Some useful relations

Modified Bessel functions,  $I_\nu(z)$  and  $K_\nu(z)$ , are defined as [24]

$$J_\nu(iz) = i^\nu I_\nu(z), \quad K_\nu(z) = (\pi/2) i^{\nu+1} H_\nu^{(1)}(iz), \quad (A.1)$$

where  $H_\nu^{(1)}(z) = J_\nu(z) + iN_\nu(z)$  is the Hankel function of the first kind. These functions for large arguments behave as

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \left( P_\nu(iz) + iQ_\nu(iz) \right), \quad |arg z| < 3\pi/2, \quad (A.2)$$

$$I_\nu(z) = \sqrt{\frac{1}{2\pi z}} e^z \left( P_\nu(iz) - iQ_\nu(iz) \right), \quad |arg z| < \pi/2, \quad (\text{A.3})$$

where

$$P_\nu(z) \sim 1 - \frac{(\mu-1)(\mu-9)}{2!(8z)^2} + \frac{(\mu-1)(\mu-9)(\mu-25)(\mu-49)}{4!(8z)^4} - \dots,$$

$$Q_\nu(z) \sim \frac{(\mu-1)}{1!(8z)} - \frac{(\mu-1)(\mu-9)(\mu-25)}{3!(8z)^3} + \dots, \quad \mu = 4\nu^2.$$

To obtain the Eq. (3.2), the following identities become important:

$$\frac{a^3}{3\pi} \int_0^\infty K^3 dK = \frac{\pi}{a} \int_0^\infty k^3 dk, \quad \text{where } k \equiv \frac{a}{\sqrt[4]{(3\pi^2)}} K,$$

$$\int_0^W k^3 dk = \sum_{l=0}^W \left( l + \frac{1}{2} \right) \sum_{n=0}^W \left( n + \frac{1}{2} \right), \quad W \text{ is an integer.}$$

Therefore one can easily extend above equation as

$$\lim_{P \rightarrow \infty} \int_0^P k^3 dk \simeq \lim_{P \rightarrow \infty} \sum_{l=0}^{l \leq P} \left( l + \frac{1}{2} \right) \sum_{n=0}^{n \leq P} \left( n + \frac{1}{2} \right), \quad P \in \mathcal{R}. \quad (\text{A.4})$$

## B Mathematical relations underling the Eq. (3.10)

Let us define  $g(z(\equiv \mathbf{z}e^{i\theta}), a)$  as

$$g(z, a) \equiv K_\nu(a \mathbf{z}e^{i(|\theta+\pi/2|-\pi/2)}), \quad \nu = l + 1/2 = 1/2, 3/2, \dots \quad (\text{B.1})$$

then it follows

1.  $g(\mathbf{z}e^{i\theta}, a) = K_\nu(a \mathbf{z}e^{i\theta}),$  for  $-\pi/2 \leq \theta \leq \pi/2,$
2.  $g(\mathbf{z}e^{i\theta_+}, a) = g(\mathbf{z}e^{i\theta_-}, a),$  where  $\theta_\pm = -\pi/2 \pm \theta',$
3.  $g(\mathbf{z}e^{i\theta}, a)$  has no zeros<sup>4</sup>, for  $-3\pi/2 \leq \theta \leq \pi/2.$

Now according to the Cauchy's theorem, it is easy to verify that we have

$$\oint_{C'} dz z \frac{d}{dz} \ln \left( \frac{g(z, a \rightarrow \infty)}{g(z, a)} \right) = 0, \quad (\text{B.2})$$

$C'$  is a counterclockwise contour of integration consists of the real axis  $[R, -R]$  and a semicircle  $C'_R$ , in the down half-plane with radius  $R$  which we let  $R$  becomes infinitely large. After separating the contributions of different parts of the counter  $C'$  and using integration by parts (see Eq. (3.7)), we can write (B.2) as

$$2 \int_0^\infty dx \ln \left( \frac{\lim_{a \rightarrow \infty} K_\nu(x a)}{K_\nu(x a)} \right) + \int_{C'_R(-\pi, 0)} z d \ln \left( \frac{g(z, a \rightarrow \infty)}{g(z, a)} \right) = 0. \quad (\text{B.3})$$

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<sup>4</sup>It has no zeros as long as  $K_\nu(z)$ , for any real number  $\nu$ , has no roots in the region  $|arg z| \leq \frac{\pi}{2}$  [24]. On the other hand note that, if we choose  $-\pi/2 \leq \theta_+ \leq \pi/2$ , the second property imposes  $g(\mathbf{z}e^{i\theta_+}, a) = g(\mathbf{z}e^{i\theta_-}, a)$ , where  $-3\pi/2 \leq \theta_- \leq -\pi/2$ , again one gets back to third property.

From (B.3), the last two terms in Eq. (3.10) turns to

$$\frac{l+1/2}{2\pi} \int_{C'_{R(-\pi,0)}} z d \ln \left( \frac{g(z,a)}{g(z,a \rightarrow \infty)} \right) + \frac{l+1/2}{2\pi i} \int_{C_{\Lambda(-\frac{\pi}{2}, \frac{\pi}{2})}} z d \ln \frac{f(z,a)}{f(z,a \rightarrow \infty)}. \quad (\text{B.4})$$

Substituting  $z \rightarrow iz$  in the last term, results in

$$\frac{l+1/2}{2\pi} \int_{C'_{R(-\pi,0)}} z d \ln \left( \frac{g(z,a)}{g(z,a \rightarrow \infty)} \right) + \frac{l+1/2}{2\pi} \int_{C_{\Lambda(-\pi,0)}} z d \ln \frac{f(iz,a)}{f(iz,a \rightarrow \infty)}, \quad (\text{B.5})$$

in which  $f(iz,a)$ , is defined as (3.6 and A.1)

$$f(iz,a) = J_\nu(a \mathbf{z} e^{i|\theta+\pi/2|}) = J_\nu(a i \mathbf{z} e^{i(|\theta+\pi/2|-\pi/2)}) = i^\nu I_\nu(a \mathbf{z} e^{i(|\theta+\pi/2|-\pi/2)}). \quad (\text{B.6})$$

Note that  $f(iz,a)$  and  $g(z,a)$  have the similar properties, therefore we can rewrite (B.5) as follows:

$$\frac{l+1/2}{\pi} \left( \int_{C'_{R(-\pi/2,0)}} z d \ln \frac{K_\nu(z a)}{\lim_{a \rightarrow \infty} K_\nu(z a)} + \int_{C_{\Lambda(-\pi/2,0)}} z d \ln \frac{I_\nu(z a)}{\lim_{a \rightarrow \infty} I_\nu(z a)} \right). \quad (\text{B.7})$$

At the large radius of the contour and after making use of (A.2), (A.3), one gets

$$(B.7) = \frac{l+1/2}{\pi} \int_{C_{\infty(-\pi/2,0)}} z d \ln (P^2(iaz) + Q^2(iaz)), \quad (\text{B.8})$$

the term under the integral falls off at large  $z$  faster than  $\frac{1}{z^2}$ , thus at this limit this term can be neglected.

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